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SOME EXACT SOLUTIONS OF A SYSTEM OF EQUATIONS OF ELECTROHYDRODYNAMICS

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A class of exact solutions of a system of equations of electrohydrodynamics is studied for which the electric current is directed along streamlines of the hydrodynamic flow. In the two-dimensional case the solution is written down explicitly. It is shown how to construct other exact solutions for which the collinearity condition of the electric current density and velocity vectors has not been satisfied, by using the solutions obtained. as an illustration, an exact solution for the flow of a unipolarly charged fluid in a channel with electrode-walls is constructed. It is shown that for a particular kind of hydrodynamic eddy current the solution of the two-dimensional system of equations can be reduced in some cases to finding the solution of a system of ordinary differential equations.

1. Let us examine the stationary flow of a unipolarly charged fluid. The parameter of the electrohydrodynamic interaction is assumed infinitesimal. A hydrodynamic stream of ideal incompressible homogeneous fluid has the potential $V^* = -\operatorname{grad} \Phi^*$. Ohm's law has the form $j^* = q^* (V^* + bE^*)$, where $b = \text{const}$ is the mobility. Let us introduce dimensionless quantities by means of formulas

$$x = l\xi,\ y = l\eta,\ z = l\zeta,\ \varphi^* = \frac{u_0 l}{b}\varphi,\ \Phi^* = u_0 l \Phi$$

$$q^* = \frac{\varepsilon_0 u_0}{4\pi b l} q \tag{1.1}$$

where φ^* is the electric field potential $E^* = -\operatorname{grad} \varphi^*$. Using the potentiality of the electric field and the velocity field, let us introduce the total potential $\chi = \Phi + \varphi$. The equations of electrohydrodynamics are [1]

$$\Delta\Phi = 0, \quad \operatorname{div}(q \operatorname{grad} \chi) = 0, \quad \Delta\chi = -q \tag{1.2}$$

Let us seek the solution of the system (1.2) for which the current density j and velocity V vectors are collinear at each point of the stream

$$\mathbf{j} = a\mathbf{V} \qquad (1.3)$$

where $a = a(\xi, \eta, \zeta)$ is an unknown function which must be found during the solution. The form of the hydrodynamic stream for which the construction of the solution (1.3) is possible will also be found during the solution.

Let us turn to the solution of the problem posed. Using Ohm's law and (1.3), we obtain an equation for a from the second equation in (1.2)

$$(\operatorname{grad} a \cdot \operatorname{grad} \Phi) = 0 \qquad (1.4)$$

If two stream functions Ψ_1 and Ψ_2 are introduced for the hydrodynamic flow [2], then we obtain from (1.4) that a is a function of Ψ_1 and Ψ_2

$$a = a(\Psi_1, \Psi_2) \qquad (1.5)$$

Taking account of Ohm's law and the potentiality of the velocity field, we find that (1.3) is equivalent to the relationship

$$\operatorname{grad} \chi = aq^{-1} \operatorname{grad} \Phi \qquad (1.6)$$

From (1.5) and (1.6) we obtain the general form of the relation between χ, q and a, Φ

$$\chi = h(\Phi), \qquad q = a(\Psi_1, \Psi_2) / h'(\Phi) \qquad (1.7)$$

where $h(\Phi)$ and $a(\Psi_1, \Psi_2)$ are the desired functions of their arguments. Substituting the expressions for χ and q from (1.7) and (1.2), we obtain equations to determine the unknown functions Φ, $h(\Phi)$ and $a(\Psi_1, \Psi_2)$ from the first and third equations of (1.2)

$$|\operatorname{grad} \Phi|^2 = -\frac{a(\Psi_1, \Psi_2)}{h'(\Phi)\, h''(\Phi)}, \qquad \Delta\Phi = 0 \qquad (1.8)$$

Therefore, the problem of finding exact solutions of the system (1.2) in the form (1.3) has been reduced to a problem for harmonic functions. Every exact solution of the problem of potential motion of an incompressible ideal fluid for which the absolute value of the velocity can be represented as the product $V = V_1(\Phi)\, V_2(\Psi_1, \Psi_2)$, generates an exact solution of the system of equations of electrohydrodynamics (1.2), where $a(\Psi_1, \Psi_2)$ and $h(\Phi)$ are found from the following relationships:

$$a(\Psi_1, \Psi_2) = -\alpha_0 V_2^2(\Psi_1, \Psi_2), \quad h(\Phi) = -\int_{\Phi_0}^{\Phi} \left[2\alpha_0 \int_{\Phi_0}^{\Phi} V_1^{-2}(\Phi)\, d\Phi + \alpha_1\right]^{1/2} d\Phi + \alpha_2$$

where α_0, α_1, α_2, Φ_0 are constants.

2. In the two-dimensional case, the problem of finding solutions of the form (1.3) for the system (1.2) can be carried out completely. In this case formulas (1.7) and (1.8) become

$$\chi = h(\Phi), \ q = \frac{a(\Psi)}{h'(\Phi)}, \quad |\operatorname{grad} \Phi|^2 = -\frac{a(\Psi)}{h'(\Phi)\, h''(\Phi)}, \quad \Delta\Phi = 0 \qquad (2.1)$$

Here Φ and Ψ agree with the potential and stream function of the potential flow of an ideal incompressible fluid taken with opposite sign, $w = \Phi + i\Psi$ is an analytical function of $\tau = \xi + i\eta$.

Let us turn to finding the functions Φ, $h(\Phi)$ and $a(\Psi)$. The function Φ is harmonic, therefore, $\ln |\mathrm{grad}\, \Phi|^2$ is also a harmonic function, hence

$$\left(\frac{\partial^2}{\partial \Phi^2} + \frac{\partial^2}{\partial \Psi^2}\right) \ln |\mathrm{grad}\, \Phi|^2 = 0 \qquad (2.2)$$

Substituting the expression for $|\mathrm{grad}\, \Phi|^2$ from (2.1) into (2.2) and taking into account that Φ and Ψ are independent, we obtain an equation for $a(\Psi)$ and $h(\Phi)$

$$\frac{\partial^2}{\partial \Phi^2} \ln |h'(\Phi) h''(\Phi)| = c, \qquad \frac{\partial^2}{\partial \Psi^2} \ln |a(\Psi)| = c, \ c = \mathrm{const} \qquad (2.3)$$

Solving (2.3), we obtain the general form for $h(\Phi)$ and $a(\Psi)$ in the two-dimensional case

$$h(\Phi) = -\int_0^\Phi \left[2c_2 \int_0^\Phi \exp(c\Phi^2 + c_1\Phi)\, d\Phi + c_5\right]^{1/2} d\Phi + c_6 \qquad (2.4)$$

$$a(\Psi) = -c_4 \exp(c\Psi^2 + c_3\Psi)$$

where $c, c_1, c_2, \ldots, c_6$ are constants. Knowing $h(\Phi)$ and $a(\Psi)$ we find an expression for $\ln |\mathrm{grad}\, \Phi|^2$ and therefore, the relation between w and τ

$$\tau - \tau_1 = \int_0^w \exp\left\{\frac{1}{2}\left[cw^2 + (c_1 + ic_3) w + c_2 - c_4 + ic_7\right]\right\} dw \qquad (2.5)$$

where $\tau_1 = \xi_1 + i\eta_1$, c_7 are constants. The electrical field potential φ and the stream function for the electric current density vector ψ ($j_\xi = \partial\psi / \partial\eta$, $j_\eta = -\partial\psi / \partial\xi$) are

$$\varphi = h(\Phi) - \Phi, \qquad \psi = -\int_0^\Psi a(\Psi)\, d\Psi + \psi_0, \qquad \psi_0 = \mathrm{const}$$

Formulas (2.4), (2.5) exhaust all solutions of the form (1.3) in the two-dimensional case. Particular solutions corresponding to one-dimensional flow ($c = c_1 = c_3 = c_7 = 0$), to the flow from a source ($c = c_3 = 0$), from a vortex ($c = c_1 = 0$), and from a vortex source ($c = 0$), have been obtained in [3]. The one-dimensional flow has also been investigated in [4].

3. We examine one general property of the exact solutions of the system (1.2). Let an arbitrary exact solution φ_*, Φ_* of the system (1.2) which has the total potential χ_* be known. In this case any pair of functions φ, Φ is also an exact solution of the system of equations of electrohydrodynamics (1.2) if two conditions are satisfied: (1) $\varphi + \Phi = \chi_*$ and (2) $\Delta\Phi = 0$. Therefore, each exact solution of the system (1.2) generates an infinite set of exact solutions φ, Φ, where the current and charge density distributions are identical for all exact solutions from this set and are determined only by the total potential χ_*.

Using the above, other exact solutions for which condition (1.3) has not been satisfied can be constructed by using the exact solutions (2.4), (2.5) obtained. Therefore, exact solutions can be constructed for problems of the flow around electrode-bodies by an electrohydrodynamic stream (the solution of this problem under an assumption on the smallness of the characteristic dimension of the electrode-body as compared with the characteristic dimension of the flow domain has been given in [5]) for the flow of a unipolarly charged fluid in a channel with electrode-walls, etc. An example of the construction of such an exact solution for the electrohydrodynamic flow in a channel with electrode-walls with the total potential $\chi = -\xi^{3/2}$ will be considered below.

Such a total potential is obtained from (2.4) if we set $c = c_1 = c_3 = c_5 = c_6 = c_7 = \xi_1 = \eta_1 = 0, \quad c_2 = c_4 = {}^9/_8$.

4. Let us consider the following problem. The total potential is given by the formula $\chi = -\xi^{3/2}, \quad \xi > 0$ in the flow domain G of a unipolarly charged fluid. The domain G is bounded by the emitter HMH′, the collectors FD and $F'D'$, and the electrode-walls $D'C'$ and DC (Fig. 1). The solid walls DE and $D'E'$ issue from the points D and D' parallel to the ξ-axis. The fluid will be without charge outside the domain $G \cup EDC \cup E'D'C'$. The hydrodynamic flow domain G' coincides with the exterior $EDC \cup E'D'C'$ It is assumed that there is no interaction between the hydrodynamic stream and the emitter and collectors [6], therefore, the hydrodynamic potential Φ can be sought in the domain G' without taking account of the influence of the grids. Symmetry relative to the ξ-axis is assumed in solving the problem, hence, it is possible to limit oneself to seeking solutions in just the lower half-plane. The point D has the coordinates $(\xi_0, -\eta_0)$. Let us limit ourselves to an analysis of the case for which $\xi_0 > {}^4/_9$. As will be shown below, this latter condition assures monotoneity of the approach CD to the ξ-axis, and therefore, also that the channel $DCC'D'$ has the shape pictired in Fig. 1. The points in the lower half-plane in the figure have the following coordinates:

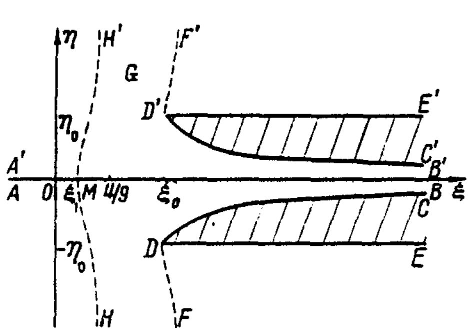

Fig. 1

$$A\,(-\infty, \quad 0), \;\; B\,(+\infty, \quad 0), \;\; C\,(+\infty, \quad -\eta^0), \;\; D\,(\xi_0, \quad -\eta_0),$$
$$E\,(+\infty, \quad -\eta_0), \;\; H\,(\xi_{1\infty}, \quad -\infty), \;\; F\,(\xi_{0\infty}, \quad -\infty), \;\; M\,(\xi_1, \quad 0).$$

Values of η^0, $\xi_{0\infty}$, $\xi_{1\infty}$ will be determined during the solution.

In the majority of cases, electrohydrodynamic inviscid flows in channels have been investigated in a one-dimensional approximation (see [7, 8], say). However, in many cases the electric forces in the flow domain of a unipolarly charged fluid turn out to be nonpotential, hence, the need arises to examine the nonuniform problem. Thus, for example, a two-dimensional problem about the motion of a unipolarly charged fluid in a channel with electrode-walls if solved in [9] by successive approximations.

In the problem considered here, the exact solution is obtained for the two-dimensional electrohydrodynamic flow in a channel with electrode-walls. The scheme to be elucidated for constructing the solution can be used to find other exact solutions if expressions are taken for χ which are different from that presented above (from (2.4), for example). The shapes of the channels will hence be different in the solution obtained.

Let us turn to the solution of the problem. It is necessary to find the shape of the electrode CD and the value of the electric potential thereon $\varphi = \varphi_0 = \text{const}$, for which the potential Φ of the hydrodynamic flow in the domain G' would yield a given total potential $\chi = -\xi^{3/2}$ in combination with φ in the domain G.

Let us assume that the desired shape CD has been found. In this case the following boundary conditions:

$$\frac{\partial\Phi(\xi,0)}{\partial\eta}=0 \quad \text{for} \quad -\infty<\xi<\infty, \quad \frac{\partial\Phi(\xi,-\eta_0)}{\partial\eta}=0 \quad \text{for} \quad \xi_0<\xi<\infty \qquad (4.1)$$

$$\Phi|_{CD}=-\xi^{3/2}+\varphi_0, \quad \frac{\partial\Phi}{\partial\xi}\to-1, \quad \frac{\partial\Phi}{\partial\eta}\to 0 \quad \text{for} \quad \xi\to-\infty, \quad \Phi(0,0)=0$$

are imposed on the function Φ in the domain G'. The function Φ is harmonic, therefore, an analytic function $w=w(\tau)=\Phi+i\Psi$ can be introduced, where $\tau=\xi+i\eta$. After insertion of the function Ψ, the boundary conditions (4.1) are equivalent to the following:

$$\Psi(\xi,0)=0 \quad \text{for} \quad -\infty<\xi<\infty, \quad \Psi(\xi,-\eta_0)=\Psi_0 \quad \text{for} \quad \xi_0<\xi<\infty$$

$$\Phi|_{CD}=-\xi^{3/2}+\varphi_0, \quad \Psi|_{CD}=\Psi_0 \qquad (4.2)$$

$$\frac{\partial\Phi}{\partial\xi}\to-1, \quad \frac{\partial\Phi}{\partial\eta}\to 0 \quad \text{for} \quad \xi\to-\infty, \quad \Phi(0,0)=0, \quad \Psi(0,0)=0$$

Here $\Psi_0=\text{const}>0$, the value of Ψ_0 is determined during the solution.

Let us turn to the question of the behavior of the curve CD. As is seen from the expression for $\Phi|_{CD}$ from (4.2), the quantity $\Phi|_{CD}$ decreases more rapidly for $\xi_0>4/9$ than in a constant stream with the potential $\Phi=-\xi$. Such a situation is possible in a channel with monotonely decreasing cross section, i.e. for CD and $C'D'$ approaching the ξ-axis monotonely. Evidently CD and $C'D'$ approach the ξ-axis asymptotically, therefore, $\eta^0=0$. Otherwise ($\eta^0\neq 0$) there would be $\Phi|_{CD}\sim k\xi$, $k=\text{const}$, for very large ξ, which results in a contradiction to (4.2).

As has already been said, $w=\Phi+i\Psi$ is an analytic function of $\tau=\xi+i\eta$, therefore, $d\tau/dw=\lambda+i\mu$ is also an analytic function. Let us turn to the determination of this function. Let us map the hydrodynamic flow domain in the lower half-plane onto the w plane, and then by using the transformation

$$w=\frac{\Psi_0}{\pi}\ln(1-\omega)+\frac{\Psi_0}{\pi}\omega+\Phi_0+i\Psi_0$$

we map the domain obtained in the w plane onto the upper half of the $\omega=\omega_1+i\omega_2$ plane. Here $\Phi_0+i\Psi_0$ is the value of w at the point $\tau=\xi_0-i\eta_0$. The images of the points A, B, C, D, E have the coordinates A_1 $(+\infty,0)$, B_1 $(1,0)$, C_1 $(1,0)$, D_1 $(0,0)$, E_1 $(-\infty,0)$ in the ω plane. Using (4.2) and the relationship between w and ω we obtain the following boundary conditions for the function $\lambda+i\mu$, which must be found in the upper half-plane:

$$\mu=0 \quad \text{for} \quad -\infty<\omega_1<0, \quad \mu=0 \quad \text{for} \quad 1<\omega_1<\infty$$

$$\lambda=-\frac{2}{3}\left(\frac{\Psi_0}{\pi}\right)^{-1/3}\left[\ln(1-\omega_1)+\omega_1+\frac{\pi\varphi_0}{\Psi_0}\right]^{-1/3} \quad \text{for} \quad 0<\omega_1<1$$

The solution of this problem is given by the Keldysh-Sedov formula in [10]

$$\lambda+i\mu=\frac{2}{3\pi}\left(\frac{\Psi_0}{\pi}\right)^{-1/3}\sqrt{\frac{\omega}{\omega-1}}\int_0^1\left[\ln(1-\varkappa)+\varkappa+\frac{\pi\varphi_0}{\Psi_0}\right]^{-1/3}\times$$

$$\sqrt{\frac{1-\varkappa}{\varkappa}}\frac{d\varkappa}{\varkappa-\omega}+\alpha\sqrt{\frac{\omega}{\omega-1}} \qquad (4.3)$$

where α is a real number.

Let us turn to the determination of the unknown constants Φ_0, Ψ_0, φ_0, α. Hence, $dw\,/\,d\tau = 1\,/\,(\lambda + i\mu) = -1$ at the point A, therefore, $\lambda + i\mu = -1$ for $\omega = +\infty$. From this condition we obtain that

$$\alpha = -1 \tag{4.4}$$

Now $dw\,/\,d\tau = \infty$ at the point B, therefore, $\lambda + i\mu = 0$ for $\omega = 1$. From this condition we obtain

$$\alpha = \frac{2}{3\pi}\left(\frac{\Psi_0}{\pi}\right)^{-1/2}\int_0^1\left[\ln(1-\varkappa)+\varkappa+\frac{\pi\varphi_0}{\Psi_0}\right]^{-1/2}\frac{d\varkappa}{\sqrt{\varkappa(1-\varkappa)}} \tag{4.5}$$

Integrating (4. 3), we obtain an expression for $\tau = \tau(\omega)$

$$\tau(\omega) = \frac{2}{3\pi}\left(\frac{\Psi_0}{\pi}\right)^{3/2}\int_0^\omega\left(\frac{\omega}{\omega-1}\right)^{1/2}d\omega\int_0^1\left[\ln(1-\varkappa)+\varkappa+\frac{\pi\varphi_0}{\Psi_0}\right]^{-1/2}\times$$
$$\sqrt{\frac{1-\varkappa}{\varkappa}}\frac{d\varkappa}{\varkappa-\omega} - \frac{\Psi_0}{\pi}\int_0^\omega\left(\frac{\omega}{\omega-1}\right)^{1/2}d\omega + a_1 + ia_2 \tag{4.6}$$

where $a_1 + ia_2$ is a constant. At the point $\tau = 0$ we have $w = 0$, hence

$$a_1 + ia_2 = -\frac{2}{3\pi}\left(\frac{\Psi_0}{\pi}\right)^{3/2}\int_0^{\omega_0}\left(\frac{\omega}{\omega-1}\right)^{1/2}d\omega\int_0^1\left[\ln(1-\varkappa)+\varkappa+\frac{\pi\varphi_0}{\Psi_0}\right]^{-1/2}\times$$
$$\sqrt{\frac{1-\varkappa}{\varkappa}}\frac{d\varkappa}{\varkappa-\omega} + \frac{\Psi_0}{\pi}\int_0^{\omega_0}\left(\frac{\omega}{\omega-1}\right)^{1/2}d\omega \tag{4.7}$$

where ω_0 is the root greater than unity of the equation $\Psi_0 \ln(\omega_0 - 1) + \Psi_0\omega_0 + \pi\Phi_0 = 0$. The point $\omega = 0$ in the ω-plane corresponds to the point $D\,(\xi_0, -\eta_0)$, therefore, we obtain from (4. 6)

$$\xi_0 - i\eta_0 = a_1 + ia_2 \tag{4.8}$$

Combining (4. 4), (4. 5), (4. 7), (4. 8), we obtain a system of transcendental equations to determine Φ_0, Ψ_0, φ_0, a_1, a_2, α as functions of ξ_0, η_0. Letting $\omega \to \omega_1 + i0$, $0 < \omega_1 < 1$ and separating real and imaginary parts in (4. 6), an equation can be obtained for the curve CD in parametric form.

Let us turn to the determination of the shape of the electrode grids and their electric potentials. The collector grids are at the potential φ_0. The emitter grid passing through the point $(\xi_1, 0)$, $0 < \xi_1 < \xi_0$, is at the potential $\varphi_1 = -\xi_1^{3/2} - \Phi(\xi_1, 0)$. We find the shape of the collectors and emitter from the condition of constancy of their electric potentials as

$$-\varphi_0 = \xi_0^{3/2} + \Phi(\xi_0, -\eta_0) = \xi_0^{3/2} + \Phi_0 = \xi^{3/2} + \Phi(\xi, \eta)$$
$$-\varphi_1 = \xi_1^{3/2} + \Phi(\xi_1, 0) = \xi^{3/2} + \Phi(\xi, \eta) \tag{4.9}$$

The values of $\xi_{0\infty}$ and $\xi_{1\infty}$ are determined from (4. 9), if it is taken into account that $\Phi(\xi, \eta) \to -\xi$ as $\eta \to \pm\infty$. The equations for $\xi_{0\infty}$ and $\xi_{1\infty}$ are

$$\xi_{0\infty}^{3/2} - \xi_{0\infty} = -\varphi_0, \quad \xi_{1\infty}^{3/2} - \xi_{1\infty} = -\varphi_1$$

The electric potential φ in the domain G is determined from the relationship

$\varphi = -\xi^{3/2} - \Phi(\xi, \eta)$; w and τ are obtained as a function of ω; these relationships yield the connection between w and τ in parametric form. The total current flowing in part of the electrode CD $(-\eta_0 < \eta < -\eta_1 \leqslant 0)$ is given by the formula $I(\eta_1) = {}^9/_8(\eta_0 - \eta_1)$. A solution can be constructed analogously if the electrodes CD and $C'D'$ are finite, the points C and C' are connected by a third collector, and solid walls go parallel to the ξ-axis from the points C and C' towards infinity (towards increasing ξ).

5. Let us examine the possibility of obtaining some exact solutions of the system of equations of electrohydrodynamics when the hydrodynamic stream is not a potential stream.

We consider a two-dimensional stationary electrohydrodynamic flow. The electrohydrodynamic interaction parameter is assumed infinitesimal. Let the hydrodynamic stream be a vortex stream of an incompressible fluid of the following form:

$$\mathbf{V}^* = (u_0 \,|\, y / l \,|^\lambda, 0), \qquad \lambda = \text{const}$$

Ohm's law has the form $\mathbf{j}^* = q^*(\mathbf{V}^* + b\mathbf{E}^*)$, $b = \text{const}$ is the mobility. We introduce dimensionless quantities analogously to (1.1). The system of equations of electrohydrodynamics [1] is

$$\operatorname{div}[q(\mathbf{V} - \operatorname{grad}\varphi)] = 0, \qquad \Delta\varphi = -q \tag{5.1}$$

where $\mathbf{V} = (|\eta|^\lambda, 0)$.

Let us seek exact solutions of the system (5.1), which can be represented as

$$\varphi = \rho^\gamma \varphi_1(\theta), \qquad q = -\rho^{\gamma-2} q_1(\theta), \qquad \rho = \sqrt{\xi^2 + \eta^2}, \qquad \theta = \operatorname{arctg}(\eta / \xi) \tag{5.2}$$

where γ is a constant. We limit ourselves to the consideration of solutions which are symmetric to the ξ-axis. In this case, the solution of the system (5.1) can be sought only for $\eta > 0$. Let us substitute (5.2) into (5.1) by first writing it in polar coordinates. The following system of equations is obtained:

$$-(\gamma - 2)\rho^{\lambda+\gamma-3} q_1 \sin^\lambda\theta\cos\theta + \rho^{\lambda+\gamma-3} q_1' \sin^{\lambda+1}\theta + \gamma(\gamma-2)\rho^{2\gamma-4} q_1\varphi_1 + \rho^{2\gamma-4} q_1'\varphi_1' + \rho^{2\gamma-4} q_1^2 = 0$$

$$\varphi_1'' + \gamma^2\varphi_1 = q_1 \tag{5.3}$$

If the exponents of ρ in the first equation of (5.3) are equated, we obtain the relation between λ and γ. It is $\gamma = \lambda + 1$. In this case the system (5.3) is transformed into a system of ordinary differential equations

$$-(\lambda - 1) q_1 \sin^\lambda\theta\cos\theta + q_1' \sin^{\lambda+1}\theta + (\lambda^2 - 1) q_1\varphi_1 + q_1'\varphi_1' + q_1^2 = 0 \tag{5.4}$$

$$\varphi_1'' + (\lambda + 1)^2\varphi_1 = q_1$$

If $\lambda > 0$, then $\mathbf{V} = 0$ for $\eta = 0$. The solution of the system (5.4) can be used in this case to study the flow of a vortex stream of unipolarly charged fluid around conductive or nonconductive walls $(\eta = 0)$. If $\lambda < 0$, then $\mathbf{V} = (\infty, 0)$ as $\eta = 0$. The solution of (5.4) in this case can be used as a model to study the distribution of the electrical parameters in a strongly vortical jet.

Let us consider the formulation of the boundary value problem for the system (5.4) in the general case. The electrohydrodynamic flow must be found in the domain G

between the emitter ($0 \leqslant \rho < \infty$, $\theta = \theta_2 = \text{const}$) and the collector ($0 \leqslant \rho < \infty$, $\theta = \theta_1 = \text{const}$). The angles θ_1 and θ_2 satisfy the inequalities $0 \leqslant \theta_1 < \theta_2 \leqslant \pi$. The uncharged fluid is charged at the emitter and discharged at the collector. The electric potential $\varphi = 0$ and the space charge $q = -q_* \rho^{\lambda-1}$, $q_* = \text{const}$ are given at the emitter. The electric potential $\varphi = 0$ is given at the collector. Therefore, the boundary conditions for the system (5.4) in the domain G are

$$\varphi_1(\theta_1) = 0, \qquad \varphi_1(\theta_2) = 0, \qquad q_1(\theta_2) = q_* \tag{5.5}$$

For $\lambda = -1$ the system (5.4) has its simplest form

$$2q_1 \operatorname{ctg}\theta + q_1' + q_1'\varphi_1' + q_1^2 = 0, \quad \varphi_1'' = q_1 \tag{5.6}$$

Let us examine this case in more detail. We eliminate q_1 from the system (5.6) by using the second equation. In place of φ_1 we introduce the new unknown function $p = 1 + \varphi_1'$. We obtain a nonlinear second-order equation for it

$$pp'' + p'^2 + 2p' \operatorname{ctg}\theta = 0 \tag{5.7}$$

Therefore, the system (5.6) admits of a reduction in order by unity. We consider the approximate solution of (5.7) when the inequalities $0 < \theta_1 < \theta_2 \ll 1$ are satisfied. Hence, $\operatorname{ctg}\theta$ in (5.7) can be replaced by θ^{-1} and its approximate solution can be sought as the exact solution of the equation

$$p_0 p_0'' + p_0'^2 + 2\theta^{-1} p_0' = 0, \quad \theta_1 \leqslant \theta \leqslant \theta_2 \tag{5.8}$$

The approximate solution is denoted by the subscript zero. The general solution of this equation is

$$\theta = \theta_1 P \exp[4k(p_0, c)], \qquad P = \left| \frac{p_0^2 - 4p_0 + c}{p_{01}^2 - 4p_{01} + c} \right|$$

$$k(p_0, c) = \int_{p_{01}}^{p_0} \frac{dp_0}{p_0^2 - 4p_0 + c} = \begin{cases} k_1(p_0, c), & c = 4 + \alpha^2 \\ k_2(p_0, c), & c = 4 \\ k_3(p_0, c), & c = 4 - \alpha^2 \end{cases}$$

$$k_1(p_0, c) = \frac{1}{\alpha}\left[\operatorname{arctg}\frac{p_0 - 2}{\alpha} - \operatorname{arctg}\frac{p_{01} - 2}{\alpha}\right], \quad k_2(p_0, c) = -\frac{1}{p_0 - 2} + \frac{1}{p_{01} - 2}$$

$$k_3(p_0, c) = \frac{1}{2\alpha} \ln \left| \frac{(p_0 - 2 - \alpha)(p_{01} - 2 + \alpha)}{(p_0 - 2 + \alpha)(p_{01} - 2 - \alpha)} \right|$$

Here c, $p_{01} = p_0(\theta_1) = \text{const}$. Assuming the boundary conditions (5.5) to be satisfied for φ_{10} and q_{10}, and also that p_0 and φ_{10} are related by $p_0 = 1 + \varphi_{10}'$, we obtain

$$\varphi_{10} = -\theta_1 P \exp[4k(p_0, c)] + \theta_1 + 2(p_0 - p_{01}) + 4\ln P + 2(8 - c)k(p_0, c)$$
$$\theta_2 = \theta_1 P \exp[4k(p_{02}, c)]$$

$$2(p_{02} - p_{01}) - 2ck(p_{02}, c) = \theta_2 - \theta_1 - 4(\ln\theta_2 - \ln\theta_1)$$

$$c = -p_{02}^2 + 4p_{02} + 2q_* p_{02}, \qquad p_{02} = p_0(\theta_2) = \text{const}$$

Here c will be greater than, equal to, or less than four depending on the values taken on by θ_1, θ_2, q_*.

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THE SPREADING OF SINGLY IONIZED JETS IN HYDRODYNAMIC STREAMS

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This investigation pertains to the spreading of singly ionized [unipolarly charged] jets in hydrodynamic streams applicable to problems of electrohydrodynamic flows downstream of a source of charged particles ("free" jets), in the channels and ducts of electrohydrodynamic systems ("enclosed" jets). Basic nondimensional parameters have been defined, upon which the intensity of spreading of the jets depends. By means of a numerical solution of the two-dimensional equations of electrohydrodynamics the distribution of the electric parameters (charge density, electric potential) in the jet and in the surrounding space has been established.